An Uniqueness Result on Spherically Stratified Media with Interior Transmission Eigenvalues

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Abstract

Given a set of transmission eigenvalues, we describe the connection between such a set and the indicator functions in entire function theory. The indicator functions control the asymptotic growth rate of the solution of the Sturm-Liouville problem which has an uniqueness in the inverse spectral theory. Accordingly, the set of transmission eigenvalues has an inverse spectral property.
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1 Introduction and the Main Result

In this paper, we consider the stationary scattering problem

$$\begin{cases} \Delta u + k^2 n(x) u = 0, & \text{in } \mathbb{R}^3; \\ u = u^s + u^i; \\ \lim_{r\to\infty} r\{\frac{\partial u^s}{\partial r} - iku^s\} = 0, \end{cases} \tag{1.1}$$

where u^i is an entire solution of the Helmholtz equation, k is the wave number and $n \in \mathcal{C}^1(0,a) \cap \mathcal{H}^2(0,a)$ is specified spherically stratified in B, where $B := \{\mathbb{R}^3 | \, |x| < a\}$, such that $n(x) = n(r) > 0$ and $n(a) = 1$. We may ask that is there any incident fields u^i such that the scattered field u^s is identically zero? The answer is positive provided that there exists a nontrivial solution to the following interior transmission problem: $k \in \mathbb{C}$, $w, v \in \mathcal{L}^2(B)$, $w - v \in \mathcal{H}_0^2(B)$,

$$\begin{cases} \Delta w + k^2 n(r) w = 0, & \text{in } B; \\ \Delta v + k^2 v = 0 & \text{in } B; \\ w = v, & \text{on } \partial B; \\ \frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} & \text{on } \partial B. \end{cases} \tag{1.2}$$

For spherical perturbations as the one given by (1.1), we consider the spherical harmonics which we refer to Colton and Kress [5] for a theory in inverse problems. In particular, we let Y_l be the solution of following equation.

$$Y_l'' + \frac{2}{r} Y_l' + \{k^2 n(r) - \frac{l(l+1)}{r^2}\} Y_l = 0, \tag{1.3}$$

such that

$$\lim_{r\to 0} \{Y_l(r) - j_l(kr)\} = 0, \tag{1.4}$$

where j_l is a spherical Bessel function. The existence of the nontrivial solution of the interior transmission problem (1.2) is implied by the existence of the nontrivial constants a_l, b_l for some k, l to the following system of equations.

$$\begin{cases} a_l Y_l(a) - b_l j_l(ka) = 0; \\ a_l Y_l'(a) - b_l k j_l'(ka) = 0. \end{cases} \tag{1.5}$$

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Equivalently, we are looking for the zeros of the following functional determinant.

$$d_l(k) := \det\begin{pmatrix} Y_l(a) & -j_l(ka) \\ Y_l'(a) & -kj_l'(ka) \end{pmatrix}. \tag{1.6}$$

These zeros of the functional determinant $d_l(k)$ are the interior transmission eigenvalues of (1.2). In particular, when $l = 0$, we consider the zeros of

$$d_0(k) := \det\begin{pmatrix} Y_0(a) & -j_0(ka) \\ Y_0'(a) & -kj_0'(ka) \end{pmatrix}, \tag{1.7}$$

where $Y_0 = \frac{y(r)}{r}$ and $y(r)$ satisfies

$$y'' + k^2 n(r)y = 0,\ y(0) = 0,\ y'(0) = 1, \tag{1.8}$$

which is well-known that there exists an unique solution to (1.8) to every $k \in \mathbb{C}$. Hence, we may see $y = y(x;k)$. $j_0(kr) = \frac{\sin kr}{kr}$. Moreover, we have the asymptotics of $y(r;k)$ and $y'(r;k)$:

$$y(x;k) = \frac{1}{[n(0)n(r)]^{\frac{1}{4}}k}\sin(k\int_0^r [n(r)]^{\frac{1}{2}}dr)[1 + O(\frac{1}{|k|})],\ \forall k \in \mathbb{C} \setminus \{0i + \mathbb{R}\}. \tag{1.9}$$

Similarly,

$$y'(x;k) = [n(r)/n(0)]^{\frac{1}{4}}\cos(k\int_0^r [n(r)]^{\frac{1}{2}}dr)[1 + O(\frac{1}{|k|})],\ \forall k \in \mathbb{C} \setminus \{0i + \mathbb{R}\}. \tag{1.10}$$

We refer such asymptotics to [1, Proposition 2.3]. Such asymptotic expansions are classic in spectral theory. See Pöschel and Trubowitz [11] and Naimark [10].

Definition 1.1 *We define* $s := a - \int_0^a [n(r)]^{\frac{1}{2}}dr$ *and* $b := \int_0^a [n(r)]^{\frac{1}{2}}dr$.

In this paper, we assume either

$$a > b \text{ or } a < b, \tag{1.11}$$

simultaneously for each $n(r)$'s. The following asymptotic behavior holds.

$$d_0(k) = \frac{1}{a^2k[n(0)]^{1/4}}\sin k(a - b) + O(\frac{1}{k^2}), \forall k \in 0i + \mathbb{R}. \tag{1.12}$$

Such asymptotic expansion is classical in spectral theory. See Colton and Kress [5]. Let us consider its behavior in $\mathbb{C}$. From (1.9) and (1.10),

$$\begin{aligned} \frac{y(a)}{a}[-kj_0'(ka)] &= [\frac{B\sin ka\sin kb}{a^3k^2} - \frac{B\cos ka\sin kb}{a^2k}][1 + O(\frac{1}{|k|})],\ B := \frac{1}{[n(0)n(a)]^{\frac{1}{4}}}; \\ j_0(ka)(\frac{y(r)}{r})'|_{r=a} &= [\frac{C\sin ka\cos kb}{a^2k} - \frac{B\sin ka\sin kb}{a^3k^2}][1 + O(\frac{1}{|k|})],\ C := [\frac{n(a)}{n(0)}]^{\frac{1}{4}}. \end{aligned} \tag{1.13}$$

Hence, we have the asymptotics for $d_0(k)$.

$$\begin{aligned} d_0(k) &= j_0(ka)(\frac{y(r)}{r})'|_{r=a} + \frac{y(a)}{a}[-kj_0'(ka)] && (1.14) \\ &= \frac{1}{a^2k[n(0)]^{\frac{1}{4}}}[\sin k(a - b)][1 + O(\frac{1}{|k|})],\ \forall k \in \mathbb{C} \setminus \{0i + \mathbb{R}\}. && (1.15) \end{aligned}$$

In Aktosun, Gintides and Papanicolaou [1, p.5], they have shown if $\int_0^a [n(r)]^{\frac{1}{2}}dr < a$, then the transmission eigenvalues corresponding to spherically symmetric solutions of the interior transmission problem uniquely determine the $n(r)$. Furthermore, in [1], it is shown if $d_0(k) \equiv 0$ for $\lambda \in \mathbb{C}$, then $n(r) \equiv 1$ in $[0, a]$. It is also discussed that the signs of the quantity $a - b$ plays a role in the inverse spectral theory in [1]. Furthermore, in Cakoni, Colton and Gintides [2, Theorem 2.1], they have shown if $n(0)$ is given and $n(r) > 1$, then $n(r)$ is uniquely determined from some knowledge of the transmission eigenvalues. It

is expected among mathematicians, say, [6], that such an uniqueness holds for the condition $n(r) > 1$. In this paper, we show that only the interior transmission eigenvalues near the real axis are needed to determine the $n(r)$. In addition, we propose another qualitative description on the counting function to the zeros of $d_l(z)$. In the spectral theory of Sturm-Liouville, such a qualitative description on the growth rate of zeros of d_l is connected to the inverse problem on finding index $n(r)$. For such a connection, we refer to McLaughlin and Polyakov [9] which is based on Pöschel and Trubowitz's work in [11]. In [9], there is an argument on the qualitative description for the zeros of $d_0(z)$.

Firstly, we need some vocabulary from entire function theory. We refer to the Levin's book [7, 8].

Definition 1.2 *Let $f(z)$ be an entire function. Let $M_f(r) := \max_{|z|=r} |f(z)|$. An entire function of $f(z)$ is said to be a function of finite order if there exists a positive constant k such that the inequality*

$$M_f(r) < e^{r^k} \tag{1.16}$$

is valid for all sufficiently large values of r. The greatest lower bound of such numbers k is called the order of the entire function $f(z)$. By the type σ of an entire function $f(z)$ of order ρ, we mean the greatest lower bound of positive number A for which asymptotically we have

$$M_f(r) < e^{Ar^\rho}. \tag{1.17}$$

That is

$$\sigma := \limsup_{r\to\infty} \frac{\ln M_f(r)}{r^\rho}. \tag{1.18}$$

If $0 < \sigma < \infty$, then we say $f(z)$ is of normal type or mean type.

Definition 1.3 *If an entire function $f(z)$ is of order one and of normal type, then we say it is an entire function of exponential type σ.*

Definition 1.4 *Let $\rho \in \mathbb{R}$ and $\rho(r) : \mathbb{R}^+ \to \mathbb{R}^+$. We say $\rho(r)$ is a proximate order to ρ if*

$$\lim_{r\to\infty} \rho(r) = \rho \geq 0;\ \lim_{r\to\infty} r\rho'(r) \ln r = 0. \tag{1.19}$$

Definition 1.5 *Let $f(z)$ be an integral function of finite order in the angle $[\theta_1, \theta_2]$, we call the following quantity as the generalized indicator of the function $f(z)$.*

$$h_f(\theta) := \limsup_{r\to\infty} \frac{|f(re^{i\theta})|}{r^{\rho(r)}},\ \theta_1 \leq \theta \leq \theta_2, \tag{1.20}$$

where $\rho(r)$ is some proximate order.

We review two inequalities for indicator functions.

$$h_{fg}(\theta) \leq h_f(\theta) + h_g(\theta); \tag{1.21}$$
$$h_{f+g} \leq \max\{h_f(\theta), h_g(\theta)\}. \tag{1.22}$$

We can find these in [7, p.51].

The order and the type of an integral function in an angle can be defined similarly. The connection between the indicator $h_f(\theta)$ and its type σ can be specified by the following theorem.

Lemma 1.6 (Levin [7], p.72) *The maximum value of the indicator $h_f(\theta)$ of the function $f(z)$ on the interval $\alpha \leq \theta \leq \beta$ is equal to the type σ_f of this function inside the angle $\alpha \leq \arg z \leq \beta$.*

Lemma 1.7 *Let $f(z) = C \sin Az$, where A, C are constants. $f(z)$ is an entire function of exponential type $|A|$.*

Proof It suffices to see $\frac{\sin z}{z} = \frac{e^{iz} - e^{-iz}}{2iz}$ and

$$h_{\frac{\sin z}{z}}(\theta) = \limsup_{r\to\infty} \frac{\ln\left|\frac{\sqrt{e^{2r\sin\theta} + e^{-2r sin\theta} - 2\cos 2(r\cos\theta)}}{2r}\right|}{r} = |sin\theta|. \tag{1.23}$$

□

For our indicator for $d_0(z)$.

Lemma 1.8 *Let $b = \int_0^a [n(r)]^{\frac{1}{2}} dr$. Then, given $d_0(z)$ as in (1.12), it is an entire function of exponential type $|a-b|$ with indicator function*

$$h_{d_0}(\theta) = |\sin\theta||a-b|. \tag{1.24}$$

Proof Let $\eta := za - zb$. We compute that

$$\begin{aligned} |\sin\eta|^2 &= \frac{1}{4}[e^{\eta+\overline{\eta}} - e^{\eta-\overline{\eta}} - e^{-\eta+\overline{\eta}} + e^{-\eta-\overline{\eta}}] \\ &= \frac{1}{4}[e^{-2ya+2yb} - e^{2ixa-2ixb} - e^{-2ixa+2ixb} + e^{2ya-2yb}] \\ &= \frac{1}{4}[e^{-2ya+2yb} + e^{2ya-2yb} - 2\cos(2xa-2xb)]. \end{aligned} \tag{1.25}$$

Following from (1.15) that

$$h_{d_0}(\theta) = \limsup_{r\to\infty} \frac{\ln|\sin\eta|}{r} = \limsup_{r\to\infty} \frac{|y||a-b|}{r} = |\sin\theta||a-b|,\ \theta \neq 0, \pi. \tag{1.26}$$

We see that $h_{d_0}(\theta)$ is continuous in $[0, 2\pi]$, Levin [7, p.54], we conclude

$$h_{d_0}(\theta) = |a-b||\sin\theta|,\ \theta \in [0, 2\pi]. \tag{1.27}$$

Furthermore, $d_0(z)$ is an entire function of exponential growth due to the definition (1.6) and the fact that spherical Bessel function $j_0(z) = \frac{\sin z}{z}$ and $y(r)$ are entire functions of exponential type.

We see that $h_{d_0}(\theta)$ attains its nontrivial maximum $|a-b|$ at $\theta = \pm\frac{\pi}{2}$. $\Box$

The following is the main theorem of this paper.

Theorem 1.9 *Let the 0-th functional determinant $d_0(z)$ be defined as above. Then, the zeros of $d_0(z)$ inside the angular wedge*

$$\Sigma_\epsilon := \{k \in \mathbb{C} |\ -\epsilon \leq \arg k \leq \epsilon,\ \pi - \epsilon \leq \arg k \leq \pi + \epsilon\},\ \forall \epsilon > 0, \tag{1.28}$$

uniquely determines between the indices $n(r)$'s provided they are of the same value at $r = 0$.

2 M.L. Cartwright's Theorem

The foundation of this paper is the following theorem in Cartwright [3, p.538][2].

Theorem 2.1 *Suppose that $f(z)$ is an integral function of order $\rho = 1$ with the following Hadamard's representation:*

$$f(z) = z^m \exp\{c_0 + c_1 z\} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp\{\frac{z}{z_n}\} \tag{2.1}$$

and that the indicator

$$h(\theta) = \max\{A\cos\theta, B\cos\theta\}, \tag{2.2}$$

where $B \geq A$, $B > 0$. Then, the following asymptotics hold.

$$\lim_{r\to\infty} \frac{n(r, -\frac{\pi}{2}+\delta, \frac{\pi}{2}-\delta)}{r^{\rho(r)}} = 0; \tag{2.3}$$

$$\lim_{r\to\infty} \frac{n(r, \frac{\pi}{2}+\delta, \frac{3\pi}{2}-\delta)}{r^{\rho(r)}} = 0; \tag{2.4}$$

$$n(r, \pm\frac{\pi}{2} - \delta, \pm\frac{\pi}{2} + \delta) \sim \frac{1}{2\pi}\{B(r) - A(r)\} r^{\rho(r)}, \tag{2.5}$$

$$c_1 + \sum_{|z_n|\leq r} \frac{1}{z_n} \sim \frac{1}{2}\{A(r) + B(r)\} r^{\rho(r)-1}, \tag{2.6}$$

where $A(r)$ and $B(r)$ are real functions such that $B(r) \geq A(r)$,

$$\liminf_{r\to\infty} A(r) = A;\ \limsup_{r\to\infty} B(r) = B; \tag{2.7}$$

$$\lim_{r\to\infty}\{A(r) - A(\eta r)\} = 0;\ \lim_{r\to\infty}\{B(r) - B(\eta r)\} = 0,\ \forall \eta > 0, \tag{2.8}$$

where $\rho(r)$ is a proximate order to ρ.

Remark 2.2 *We may choose $A = -B$ for this note. We also note that $d_0(z)$ is bounded on $0i + \mathbb{R}$. Cartwright's paper may be hard to obtained. We consider the Levin's book on functions of class C in [8] for some modern reference and backup theory source. To answer some mathematicians' question, rotating the functional $d(z)$ by $\frac{\pi}{2}$ won't alter the nature of the distribution of the zeros. Such a picture is clear in the discussion in Levin [8, p.127]. Our functional $d_l(z)$ is trivially of class C as already discussed in Chen [4]. Two conditions listed in [8, p.115] can be justified by the fact that $d_l(z)$ is bounded over the real axis. We compare with the indicator function appears in (2.2), the one in Lemma 1.8 and the one in Levin [8, p.126]. We see all these three functions are consistent with each other. Another examination on (3.5), (3.8), (2.6) and (3.17) yields the same indicator functions.*

Corollary 2.3 *Let $n(d^j, r, \alpha, \beta)$ be the number of zeros of $d^j(z)$ that are located in the closed cone $\{z \in \mathbb{C} | \alpha \leq \arg z \leq \beta,\ |z| \leq r\}$ and*

$$\Delta^j(\alpha, \beta) := \lim_{r\to\infty} \frac{n(d^j, r, \alpha, \beta)}{r};\ \Delta^j(\beta) - \Delta(\alpha) := \Delta(\alpha, \beta) + C, \tag{2.9}$$

where $j = 1, 2$, and C is some constant. Then,

$$\Delta^j(\delta, \pi - \delta) = 0; \tag{2.10}$$
$$\Delta^j(\pi + \delta, -\delta) = 0; \tag{2.11}$$
$$\Delta^j(-\delta, \delta) > 0; \tag{2.12}$$
$$\Delta^j(\pi - \delta, \pi + \delta) > 0; \tag{2.13}$$
$$\Delta^1(-\delta, \delta) = \Delta^2(-\delta, \delta) \neq 0; \tag{2.14}$$
$$\Delta^1(\pi - \delta, \pi + \delta) = \Delta^2(\pi - \delta, \pi + \delta) \neq 0. \tag{2.15}$$

Accordingly, let E be the set of points of discontinuity of the function $\Delta^j(\psi)$. Then, $E = \{0, \pi\}$.

Proof This is only Cartwright's theory. The indicator function $h_{d^j}(\theta)$ is computed in Lemma 1.8. Combining with (2.2), we prove the corollary. □

3 Proof of Theorem 1.9

Let $d(z) := d_0(z)$. From (2.1), suppose that

$$d(z) = z^m \exp\{c_0 + c_1 z\} \prod_{n=1}^{\infty} (1 - \frac{z}{z_n}) \exp\{\frac{z}{z_n}\}. \tag{3.1}$$

Letting $z := re^{i\theta}$.

$$\delta(r) := c_1 + \sum_{|z_n|\leq r} \frac{1}{z_n}. \tag{3.2}$$

$$f_r(z) := \prod_{|z_n|\leq r} (1 - \frac{z}{z_n}) \prod_{|z_n|>r} (1 - \frac{z}{z_n}) e^{\frac{z}{z_n}}. \tag{3.3}$$

For each index $n^j(r)$, we use

$$f_r^j(z) := \prod_{|z_n^j|\leq r} (1 - \frac{z}{z_n^j}) \prod_{|z_n^j|>r} (1 - \frac{z}{z_n^j}) e^{\frac{z}{z_n^j}}. \tag{3.4}$$

Hence, (3.1) becomes

$$d(z) = z^m e^{c_0} e^{\delta(r)z} f_r(z). \tag{3.5}$$

From (3.5), we have

$$h_d(\theta) = \limsup_{r\to\infty} \frac{\ln|d(re^{i\theta})|}{r} = \limsup_{r\to\infty} \frac{|\delta(r)z|}{r} + \limsup_{r\to\infty} \frac{\ln|f_r(re^{i\theta})|}{r}. \tag{3.6}$$

For the first limit in (3.6). We see from (2.6) that $\delta(r) \sim 0$. That is $|\delta(r)| < \delta'$ for some large r for any given $\delta' > 0$. Hence,

$$|e^{\delta(r)z}| < e^{|\delta(r)z|} = e^{|\delta(r)|r} < e^{\delta' r}, \forall \delta' > 0. \tag{3.7}$$

Most important of all, we approximate the infinite product $f_r(z)$ by Levin [7, p.112. Lemma 5; p.92. Theorem 2]: out of some zero density set we have the following asymptotic behavior:

$$\ln|f_r(re^{i\theta})| \sim H_1(\theta) r^{\rho(r)}, \text{ where } \rho(r) \to 1, \tag{3.8}$$

in which

$$H_1(\theta) = -\int_{\theta-2\pi}^{\theta} [(\theta - \psi - \pi)\sin(\theta - \psi) + \cos(\theta - \psi)] d\Delta(\psi). \tag{3.9}$$

That integral is approximated by the following sum:

$$S_n(\theta) = -\sum_{l=0}^{n} [(\theta - \psi_l - \pi)\sin(\theta - \psi_l) + \cos(\theta - \psi_l)][\Delta(\psi_{l+1}) - \Delta(\psi_l)], \tag{3.10}$$

where $\psi_0 < \psi_1 < \cdots < \psi_n < \psi_{n+1}$, $|\psi_{l+1} - \psi_l| < \delta$, $l = 0, 1, 2, \cdots, n$. $\psi_{n+1} = \psi_0 + 2\pi$. We can choose δ small such that

$$|H_1(\theta) - S_n(\theta)| < \epsilon/3. \tag{3.11}$$

Let $h^j(\theta)$, $H_1^j(\theta)$, $\{\psi_k^j\}$, $f_r^j(z)$ and $S_n^j(\theta)$ be the corresponding quantities with respect to index n^j. Let $0 \in [\psi_{l_0}^1, \psi_{l_0+1}^1]$, $0 \in [\psi_{l_0}^2, \psi_{l_0+1}^2]$, $\pi \in [\psi_{l_\pi}^1, \psi_{l_\pi+1}^1]$, $\pi \in [\psi_{l_\pi}^2, \psi_{l_\pi+1}^2]$ be the angular intervals containing nonzero density angles which are 0 and π by (2.14) and (2.15). Using (2.10) and (2.11),

$$\begin{aligned} S_n^1(\theta) - S_n^2(\theta) &= -[(\theta - \psi_{l_0}^1 - \pi)\sin(\theta - \psi_{l_0}^1) + \cos(\theta - \psi_{l_0}^1)][\Delta(\psi_{l_0+1}^1) - \Delta^1(\psi_{l_0})] && (3.12) \\ &\quad -[(\theta - \psi_{l_\pi}^1 - \pi)\sin(\theta - \psi_{l_\pi}^1) + \cos(\theta - \psi_{l_\pi}^1)][\Delta(\psi_{l_\pi+1}^1) - \Delta^1(\psi_{l_\pi})] && (3.13) \\ &\quad - \text{similar terms from } S^2(\theta). && (3.14) \end{aligned}$$

Using (2.14) and (2.15), we conclude that, $\forall n$,

$$S_n^1(\theta) \equiv S_n^2(\theta). \tag{3.15}$$

Accordingly,

$$H_1^1(\theta) \equiv H_1^2(\theta). \tag{3.16}$$

Similarly, (3.9) gives

$$H_1^j(\theta) = \pi\Delta^j |\sin\theta|, \tag{3.17}$$

with its maximal value happening at $\theta = \pm\frac{\pi}{2}$. Also,

$$\ln|f_r^1(re^{i\theta})| \sim \ln|f_r^2(re^{i\theta})|, \text{ as } r \to \infty. \tag{3.18}$$

We conclude from (3.6), (3.7, (3.8), (3.16) and (3.18),

$$h^1(\theta) \equiv h^2(\theta). \tag{3.19}$$

Hence, we proved the following theorem.

Proposition 3.1 *If zeros of $d^j(z)$, $j = 1, 2$ inside the wedge Σ_ϵ, $\forall \epsilon > 0$ shares the same density, then the indicator function $h^1(\theta) \equiv h^2(\theta)$ in $[0, \pi]$. In particular, d^1 and d^2 are entire functions of exponential type of the same type $|a - \int_0^a \sqrt{n^1(r)}dr| = |a - \int_0^a \sqrt{n^2(r)}dr|$.*

Proof Since they share the same indicator function, the related maximal value of the indicator functions is the same. Hence, d^1, d^2 are entire functions of the same type by Lemma 1.8. The type is just $|a - \int_0^a \sqrt{n^1(r)}dr| = |a - \int_0^a \sqrt{n^2(r)}dr|$. $\square$

Definition 3.2 *Let* $b^j := \int_0^a \sqrt{n^j(r)}dr$. $s^j := a - b^j$, $j = 1, 2$.

Corollary 3.3 *Let* $h_{d^1-d^2}(\theta)$ *be the indicator function related to the* $d^1(z) - d^2(z)$. *Then,*

$$h_{d^1-d^2}(\pm\frac{\pi}{2}) = 0. \tag{3.20}$$

Proof By definition and that $n^1(0) = n^2(0)$, $s^1 = s^2$, (1.11) and (1.15) implies

$$h_{d^1-d^2}(\theta) = \limsup_{r\to\infty} \frac{\ln|d^1(z) - d^2(z)|}{r} = 0. \tag{3.21}$$

$\square$

On the other hand, we consider the substraction in the form of (3.5). Let $f_r^j(z)$, $j = 1, 2$, be the quantities corresponding to $n^j(r)$ in (3.3). We set

$$\begin{aligned} f_r^j(z) &:= \prod_{\{|z_n^j|\leq r\}}(1-\frac{z}{z_n^j})\prod_{\{|z_n^j|>r\}}(1-\frac{z}{z_n^j})e^{\frac{z}{z_n^j}} && (3.22)\\ &:= (\prod_{\{|z_n^j|\leq r;\, z_n^j\in\Sigma_\epsilon\}}\prod_{\{|z_n^j|\leq r;\, z_n^j\notin\Sigma_\epsilon\}})(1-\frac{z}{z_n^j})(\prod_{\{|z_n^j|>r;\, z_n^j\in\Sigma_\epsilon\}}\prod_{\{|z_n^j|>r;\, z_n^j\notin\Sigma_\epsilon\}})(1-\frac{z}{z_n^j})e^{\frac{z}{z_n^j}}. && (3.23)\end{aligned}$$

Using the assumption that zeros inside Σ_ϵ coincides as a set, we have

$$\begin{aligned} d^1(z) - d^2(z) &= z^{m^1}\prod_{|z_n^1|\leq r;\, z_n^1\in\Sigma_\epsilon}(1-\frac{z}{z_n^1})\prod_{|z_n^1|>r;\, z_n^1\in\Sigma_\epsilon}(1-\frac{z}{z_n^1})e^{\frac{z}{z_n^1}}\\ &\quad\times[e^{c_0^1}e^{\delta^1(r)z}\prod_{|z_n^1|\leq r;\, z_n^1\notin\Sigma_\epsilon}(1-\frac{z}{z_n^1})\prod_{|z_n^1|>r;\, z_n^1\notin\Sigma_\epsilon}(1-\frac{z}{z_n^1})e^{\frac{z}{z_n^1}}\\ &\quad-e^{c_0^2}e^{\delta^2(r)z}\prod_{|z_n^2|\leq r;\, z_n^2\notin\Sigma_\epsilon}(1-\frac{z}{z_n^2})\prod_{|z_n^2|>r;\, z_n^2\notin\Sigma_\epsilon}(1-\frac{z}{z_n^2})e^{\frac{z}{z_n^2}}]. \end{aligned} \tag{3.24}$$

Let us define

$$\begin{aligned} Q_\epsilon(z) &:= z^{m^1}[e^{c_0^1}e^{\delta^1(r)z}\prod_{|z_n^1|\leq r;\, z_n^1\notin\Sigma_\epsilon}(1-\frac{z}{z_n^1})\prod_{|z_n^1|>r;\, z_n^1\notin\Sigma_\epsilon}(1-\frac{z}{z_n^1})e^{\frac{z}{z_n^1}}\\ &\quad-e^{c_0^2}e^{\delta^2(r)z}\prod_{|z_n^2|\leq r;\, z_n^2\notin\Sigma_\epsilon}(1-\frac{z}{z_n^2})\prod_{|z_n^2|>r;\, z_n^2\notin\Sigma_\epsilon}(1-\frac{z}{z_n^2})e^{\frac{z}{z_n^2}}], \end{aligned} \tag{3.25}$$

which has no zero along $0i + \mathbb{R}$. Now we have

$$d^1(z) - d^2(z) = [\prod_{|z_n^1|\leq r;\, z_n^1\in\Sigma_\epsilon}(1-\frac{z}{z_n^1})\prod_{|z_n^1|>r;\, z_n^1\in\Sigma_\epsilon}(1-\frac{z}{z_n^1})e^{\frac{z}{z_n^1}}]\, Q_\epsilon(z). \tag{3.26}$$

Using (2.6),

$$|e^{\delta^j(r)z}| \lesssim Ce^{\delta|z|},\ \forall\delta > 0,\ j = 1, 2. \tag{3.27}$$

Given $\{z_n^j\}$ of zero density outside Σ_ϵ, we compute the following quantity.

$$\overline{f}_r^j(z) := \prod_{|z_n^j|\leq r;\, z_n^j\notin\Sigma_\epsilon}(1-\frac{z}{z_n^j})\prod_{|z_n^j|>r;\, z_n^j\notin\Sigma_\epsilon}(1-\frac{z}{z_n^j})e^{\frac{z}{z_n^j}}. \tag{3.28}$$

We use formula (3.8) and (3.9).

$$\ln|\overline{f}_r^j(re^{i\theta})| \sim \overline{H}_1^j(\theta)r, \text{ where } r\to\infty, \tag{3.29}$$

in which

$$\overline{H}_1^j(\theta) = -\int_{\theta-2\pi}^{\theta} [(\theta-\psi-\pi)\sin(\theta-\psi)+\cos(\theta-\psi)]d\Delta^j(\psi), \tag{3.30}$$

outside certain exceptional set. We refer to [7, p.112 Lemma 5] for a complete introduction. For the application here, we obtain that $\Delta^j(\psi) \equiv 0$ for zeros outside Σ_ϵ. That is

$$\overline{H}_1^j(\theta) \equiv 0,\ j = 1, 2. \tag{3.31}$$

We note that $\overline{H}_1^j(\theta) = h_{\overline{f}_r^j}(\theta)$, the indicator function of $\overline{f}_r^j$, $j = 1, 2$. We refer this to [7, p.498].

Using (1.21) and (1.22), we have

$$h_{Q_\epsilon}(\theta) \leq \max\{h_{e^{\delta^1(r)z}\overline{f}_r^1(z)},\ h_{e^{\delta^2(r)z}\overline{f}_r^2(z)}\}. \tag{3.32}$$

Therefore, (3.27), (3.29) and (3.31) combine to yield $h_{Q_\epsilon}(\theta) = 0$ and

$$|Q_\epsilon(z)| \lesssim Ce^{\delta|z|},\ \text{ for some } C > 0,\ \forall\delta > 0. \tag{3.33}$$

Again, the infinite product $[\prod_{|z_n^1|\leq r;\, z_n^1\in\Sigma_\epsilon}(1-\frac{z}{z_n^1})\prod_{|z_n^1|>r;\, z_n^1\in\Sigma_\epsilon}(1-\frac{z}{z_n^1})e^{\frac{z}{z_n^1}}]$ in (3.26) can be computed similarly as the $f_r(z)$ in the (3.3), (3.5) and (3.6). From (3.6), (3.8), (3.9), (3.17) and Lindelöf's theorem [7, p.28], we have

$$h_{d^1-d^2}(\theta) = \pi\Delta^1(-\delta,\delta)|\sin\theta|. \tag{3.34}$$

However, Corollary 3.3 says that $\Delta^1(-\delta,\delta) = 0$. Hence, $d^1(z) - d^2(z)$ is an exponential function of minimal type.

Therefore, (3.33) and (3.34) give

$$|d^1(z) - d^2(z)| \lesssim Ce^{\delta|z|},\ \forall\delta > 0. \tag{3.35}$$

In addition to that, using (1.12),

$$d^1(x) - d^2(x) \to 0,\ \text{ as } x \to 0i \pm \infty. \tag{3.36}$$

Using Phragmén-Lindelöf theorem as in Titchmarsh [12, p.178], (3.35) and (3.36) imply that $d^1(z)-d^2(z)$ is bounded both in lower and half complex planes. Any bounded entire function is a constant which is zero as suggested by (3.36). Hence,

$$d^1(z) \equiv d^2(z). \tag{3.37}$$

Finally, following the argument of Aktosun, Gintides and Papanicolaou around (3.7), (3.8), (3.9), (3.10) in [1, Corollary 2.10], we see that

$$d(z) = \frac{1}{a^2}\{\frac{\sin za}{z}y'(a) - \cos(za)y(a)\}. \tag{3.38}$$

Consider a substraction for two different $n^j(r)$'s, we obtain

$$d^1(z) - d^2(z) = \frac{1}{a^2}\{\frac{\sin za}{z}[y^{1\prime}(a) - y^{2\prime}(a)] - \cos(za)[y^1(a) - y^2(a)]\} \equiv 0. \tag{3.39}$$

Let $za = n\pi$, $n \in \mathbb{Z}$. In this case,

$$y^1(a;\frac{n\pi}{a}) = y^2(a;\frac{n\pi}{a}),\ n \in \mathbb{Z}. \tag{3.40}$$

Similarly, let $za = \frac{n\pi}{2}$, n, odd. We obtain

$$y^{1\prime}(a;\frac{n\pi}{2a}) = y^{2\prime}(a;\frac{n\pi}{2a}),\ n \text{ odd }. \tag{3.41}$$

Again, $y^j(a;z)$'s are entire functions of exponential type. We use a generalized Carlson's theorem from Levin [7, p.190]. This is a substitute for Phragmén-Lindelöf theorem.

Theorem 3.4 *Let $F(z)$ be holomorphic and at most of normal type with respect to the proximate order $\rho(r)$ in the angle $\alpha \leq \arg z \leq \alpha + \pi/\rho$ and vanish on a set $N := \{a_k\}$ in this angle, with angular density $\Delta_N(\psi)$. Let*

$$H_N(\theta) := \pi \int_\alpha^{\alpha+\pi/\rho} \sin|\psi - \theta| d\Delta_N(\psi),$$

when ρ is integral. Then, if $F(z)$ is not identically zero,

$$h_F(\alpha) + h_F(\alpha + \pi/\rho) \geq H_N(\alpha) + H_N(\alpha + \pi/\rho). \tag{3.42}$$

Now let

$$F(z) := y^1(a;z) - y^2(a;z), \tag{3.43}$$

$$G(z) := {y^1}'(a;z) - {y^2}'(a;z) \tag{3.44}$$

and $\rho \equiv 1$, $\alpha = -\frac{\pi}{2}$.

We deal with $F(z)$ firstly, it has a set of common zeros of density Δ^N and supported only at $\psi = 0, \pi$. Hence,

$$H_N(\theta) = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin|\psi - \theta| d\Delta_N(\psi) = \Delta^N \pi |\sin\theta|,\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \tag{3.45}$$

where N is the set of common zeros as described by (3.40). Accordingly, we may compute from (3.40) that $\Delta^N = \frac{a}{\pi}$. We refer to [7, p.91; Ch 2. Sec 3.] for a systematic and a step by step calculation. Besides that, we need to compute the indicator function $h_F(\theta)$. Recalling from (1.9),

$$y^j(a;z) = \frac{1}{[\epsilon_1^j(0)]^{\frac{1}{4}} z} \sin[z \int_0^a \sqrt{\epsilon_1^j(\rho)} d\rho][1 + O(\frac{1}{z})],\ j = 1, 2. \tag{3.46}$$

Therefore,

$$\ln|y^1(a;z) - y^2(a;z)| = \ln|\frac{1}{z[\epsilon_1^1(0)]^{\frac{1}{4}}}| + \ln|\sin[z \int_0^a \sqrt{\epsilon_1^1(\rho)} d\rho] - \sin[z \int_0^a \sqrt{\epsilon_1^2(\rho)} d\rho]| + \ln|1 + O(\frac{1}{z})|.$$

Using (1.9), (1.11), Proposition 3.1 and Definition 1.5, we obtain

$$h_F(\theta) = 0,\ \theta \neq 0,\ \pi. \tag{3.47}$$

Again, $h_F(\theta)$ is a continuous function given F is entire. Hence, we have

$$h_F(\theta) = 0,\ \theta \in [0, 2\pi]. \tag{3.48}$$

Combining Theorem 3.4, (3.45) and (3.48), we obtain $F(z) \equiv 0$.

Similarly, we can prove $G(z) \equiv 0$. In particular, we have shown

$$y^1(a;z) \equiv y^2(a;z);\ {y^1}'(a;z) \equiv {y^2}'(a;z). \tag{3.49}$$

The last ingredient is applying the uniqueness for the following inverse Sturm-Liouville problem

$$\psi''(x) + k^2 \rho(x) \psi(x) = 0,\ 0 < x < a; \tag{3.50}$$

$$\psi(0) = \psi(a) = 0. \tag{3.51}$$

as did in [1, Corollary 2.10]. We conclude that

$$\epsilon_1^1(r) \equiv \epsilon_1^2(r). \tag{3.52}$$

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